5.5 Prisms

Prisms play many different roles in Optics; there are prism combinations that serve as beamsplitters (p. 127), polarizing devices (see Section 8.4.3), and interferometers. Despite this diversity, the vast majority of applications make use of only one of two main prism functions. First, a prism can serve as a dispersive device, as it does in a variety of spectrum analyzers (p. 187). As such it is capable of separating, to some extent,

the constituent frequency components in a polychromatic light beam. Recall that the term *dispersion* was introduced earlier (p. 188) in connection with the frequency dependence of the index of refraction, $n(\omega)$, for dielectrics. In fact, the prism provides a highly useful means of measuring $n(\omega)$ over a wide range of frequencies and for a variety of materials (including gases and liquids).

Its second and more common function is to effect a change in the orientation of an image or in the direction of propagation of a beam. Prisms are incorporated in many optical instruments, often simply to fold the system into a confined space. There are inversion prisms, reversion prisms, and prisms that deviate a beam without inversion or reversion—and all of this without dispersion.

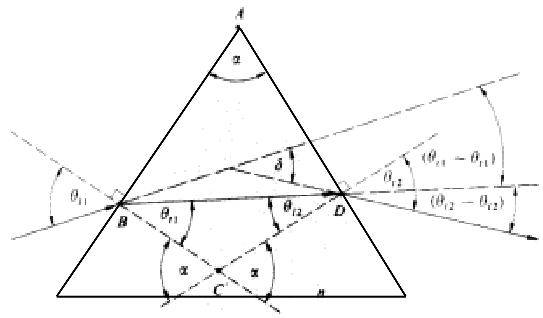


Figure 5.56 Geometry of a dispersing prism.

5.5.1 Dispersing Prisms

Prisms come in many sizes and shapes and perform a variety of functions (see photo). Let's first consider the group known as **dispersing prisms**. Typically, a ray entering a dispersing prism, as in Fig. 5.56, will emerge having been deflected from its original direction by an angle δ known as the **angular deviation**. At the first refraction the ray is deviated through an angle $(\theta_{i1} - \theta_{t1})$, and at the second refraction it is further deflected through $(\theta_{t2} - \theta_{i2})$. The total deviation is then

$$\delta = (\theta_{i1} - \theta_{t1}) + (\theta_{t2} - \theta_{i2})$$

Since the polygon $ABCD$ contains two right angles, $\measuredangle BCD$ must be the supplement of the **apex angle** α. As the exterior angle to triangle BCD, α is also the sum of the alternate interior angles, that is,

$$\alpha = \theta_{t1} + \theta_{i2} \tag{5.51}$$

Thus

$$\delta = \theta_{i1} + \theta_{t2} - \alpha \tag{5.52}$$

We would like to write δ as a function of both the angle-of-incidence for the ray (i.e., θ_{i1}) and the prism angle α; these presumably would be known. If the prism index is n and it's immersed in air ($n_a \approx 1$), it follows from Snell's Law that

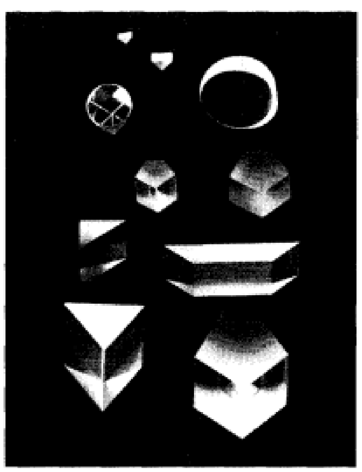
$$\theta_{t2} = \sin^{-1}(n \sin \theta_{i2}) = \sin^{-1}[n \sin(\alpha - \theta_{t1})]$$

Upon expanding this expression, replacing $\cos\theta_{t1}$ by $(1 - \sin^2\theta_{t1})^{1/2}$, and using Snell's Law we have

$$\theta_{t2} = \sin^{-1}[(\sin\alpha)(n^2 - \sin^2\theta_{i1})^{1/2} - \sin\theta_{i1}\cos\alpha]$$

The deviation is then

$$\delta = \theta_{i1} + \sin^{-1}[(\sin\alpha)(n^2 - \sin^2\theta_{i1})^{1/2} - \sin\theta_{i1}\cos\alpha] - \alpha \tag{5.53}$$



Prisms. (Photo courtesy Melles Griot.)

Apparently, δ increases with n, which is itself a function of frequency, so we might designate the deviation as $\delta(\nu)$ or $\delta(\lambda)$. For most transparent dielectrics of practical concern, $n(\lambda)$ decreases as the wavelength increases across the visible [refer back to Fig. 3.41 for a plot of $n(\lambda)$ versus λ for various glasses]. Clearly, then, $\delta(\lambda)$ will be less for red light than it is for blue.

Missionary reports from Asia in the early 1600s indicated that prisms were well known and highly valued in China because of their ability to generate color. A number of scientists of the era, particularly Marci, Grimaldi, and Boyle, had made some observations using prisms, but it remained for the great Sir Isaac Newton to perform the first definitive studies of dispersion. On February 6, 1672, Newton presented a classic paper to the Royal Society entitled "A New Theory about Light and Colours." He had concluded that white light consisted of a mixture of various colors and that the process of refraction was color-dependent.

Returning to Eq. (5.53), it's evident that the deviation suffered by a monochromatic beam on traversing a given prism (i.e., n and α are fixed) is a function only of the incident angle at the first face, θ_{i1}. A plot of the results of Eq. (5.53) as applied to a typical glass prism is shown in Fig. 5.57. The smallest value of δ is known as the **minimum deviation**, δ_m, and it is of particular interest for practical reasons. The value of δ_m can be determined analytically by differentiating Eq. (5.53) and then setting $d\delta/d\theta_{i1} = 0$, but a more indirect route will certainly be simpler. Differentiating Eq. (5.52) and setting it equal to zero yields

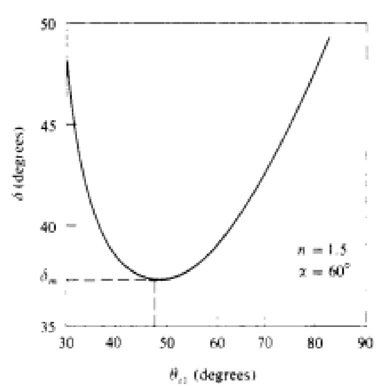


Figure 5.57 Deviation versus incident angle.

$$\frac{d\delta}{d\theta_{i1}} = 1 + \frac{d\theta_{t2}}{d\theta_{i1}} = 0$$

or $d\theta_{t2}/d\theta_{i1} = -1$. Taking the derivative of Snell's Law at each interface, we get

$$\cos\theta_{i1}\, d\theta_{i1} = n\cos\theta_{t1}\, d\theta_{t1}$$

and
$$\cos\theta_{t2}\, d\theta_{t2} = n\cos\theta_{i2}\, d\theta_{i2}$$

Note as well, on differentiating Eq. (5.51), that $d\theta_{t1} = -d\theta_{i2}$, since $d\alpha = 0$. Dividing the last two equations and substituting for the derivatives leads to

$$\frac{\cos\theta_{i1}}{\cos\theta_{t2}} = \frac{\cos\theta_{t1}}{\cos\theta_{i2}}$$

Making use of Snell's Law once again, we can rewrite this as

$$\frac{1-\sin^2\theta_{i1}}{1-\sin^2\theta_{t2}} = \frac{n^2-\sin^2\theta_{i1}}{n^2-\sin^2\theta_{t2}}$$

The value of θ_{i1} for which this is true is the one for which $d\delta/d\theta_{i1} = 0$. Inasmuch as $n \neq 1$, it follows that

$$\theta_{i1} = \theta_{t2}$$

and therefore

$$\theta_{t1} = \theta_{i2}$$

This means that *the ray for which the deviation is a minimum traverses the prism symmetrically, that is, parallel to its base.* Incidentally, there is a lovely argument for why θ_{i1} must equal θ_{t2}, which is neither as mathematical nor as tedious as the one we have evolved. In brief, suppose a ray undergoes a minimum deviation and $\theta_{i1} \neq \theta_{t2}$. Then if we reverse the ray, it will retrace the same path, so δ must be unchanged (i.e., $\delta = \delta_m$). But this implies that there are two different incident angles for which the deviation is a minimum, and this we know is not true—ergo $\theta_{i1} = \theta_{t2}$.

In the case when $\delta = \delta_m$, it follows from Eqs. (5.51) and (5.52) that $\theta_{i1} = (\delta_m + \alpha)/2$ and $\theta_{t1} = \alpha/2$, whereupon Snell's Law at the first interface leads to

$$n = \frac{\sin[(\delta_m + \alpha)/2]}{\sin\alpha/2} \tag{5.54}$$

This equation forms the basis of one of the most accurate techniques for determining the refractive index of a transparent substance. Effectively, one fashions a prism out of the material in question, and then, measuring α and $\delta_m(\lambda)$, $n(\lambda)$ is computed employing Eq. (5.54) at each wavelength of interest. Hollow prisms whose sides are fabricated of plane-parallel glass can be filled with liquids or gases under high pressure; the glass plates will not result in any deviation of their own.